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FLOW IN SLOWLY VARYING MICROCHANNELS OF RECTANGULAR CROSS-SECTION

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ABSTRACT

Laminar fully developed flow in streamwise-periodic microchannels of rectangular cross-section has been studied in this work. Based on the lubrication approximation, an analytical approach is proposed for the frictional flow resistance. Using the analytical model developed for straight channels of arbitrary cross-section, compact models are developed for linear and sinusoidal wall shapes. The models are then examined with two asymptotic two-dimensional flows of: (a) very narrow and (b) very tall channels and compared with the available experimental and numerical data in the literature. An independent numerical study is also performed to evaluate the proposed models for the three-dimensional flow case. Effects of the channel aspect ratio and deviation factor on the pressure drop are also investigated. Results for the three-simensional flow show that the proposed compact models capture the trend of the numerical data with good accuracy.

Keywords: slowly varying cross-section, fully developed flow, lubrication approximation.

Introduction

There are numerous instances of channels which have streamwise-periodic cross-sections [1-5]. It has been experimentally and numerically observed that the entrance lengths of fluid flow and heat transfer for such streamwise-periodic ducts are much shorter than those of plain ducts, and quite often, three to five cycles can make both the flow and heat transfer fully-developed [6]. In engineering practice the streamwise length of these ducts is usually much longer than several cycles; therefore theoretical works for such ducts often focus on the periodically fully-developed fluid flow and heat transfer. Rough tubes or channels with ribs on their surfaces are examples of streamwise-periodic ducts which are widely used in the cooling of electronic equipment and gas turbine blades, as well as high performance heat exchangers [5-7].

In microsystems, microchannels with converging-diverging sections maybe fabricated to influence cross-stream mixing [8-14]. Theoretically, decreasing the mixing length and producing three-dimensional velocities result in a more efficient mixer. The simplest potential design for a steady-state mixer in low Reynolds number regimes is that of a channel with variations of shape which include changes of both the curvature and the cross-sectional dimensions in the streamwise direction [14]. Also this type of channels can be found as a result of fabrication processes such as micromachining or soft lithography [15].

Several researchers investigated the flow and heat transfer in streamwise-periodic wavy channels. Some of the previous studies have been listed in Table 1. Sinusoidal wall shape is common with a range of Reynolds numbers that covers both laminar and turbulent flows.

Table 1. Some of the previous works on varying-cross section channels

author	year	wall shape	Re	method	pressure drop	heat transfer
Sparrow & Prata [6]	1983	linear	100-1000	N/E	✓	✓
Wang and Vanka [16]	1995	sinusoidal	5-800	N	✓	✓
Russ and Beer [17,18]	1996-7	sinusoidal	400-2000	N/E	✓	✓
Niceno and Nobile [19]	2000	sinusoidal/arc-shape	15-500	N	✓	✓
Wang and Chen [20]	2002	sinusoidal	100-700	N	✓	✓
Mahmud et al. [21]	2002	sinusoidal	100-2000	N	✓	
Stalio and Piller [22]	2003	sinusoidal		N	✓	✓
Nishimura et al. [23]	2003	sinusoidal	50-1000	E	✓	
Bahaidarah [24]	2005	sinusoidal /arc-shape	25-400	N	✓	✓
Naphon [25]	2007	linear	500-1400	E	✓	✓

N = numerical, E = experimental

General objectives of the above mentioned studies are to predict the transition from laminar to turbulent flow, heat transfer enhancement, and pressure drop in periodically varying cross-section channels.

In microscale applications however, the Reynolds number is low (typically less than 10) and instabilities in the flow do not generally occur [14]. The simplest solution for the flow of a Newtonian fluid through a slowly varying cross-section channel is to approximate the flow by assuming Hagen-Poiseuille flow at each axial location along the channel; this is known as the lubrication approximation [26]. Several papers can be found in the literature using this approximation to predict the pressure drop of the flow in two-dimensional varying cross-sectional channels [27-31].

Although good results can be found for gradually varying channels using the lubrication approximation, this method is not very accurate for highly constricted channels. More accurate results can be found by using the perturbation technique [32]. In this method, variation of the channel wall is transformed from the boundary condition into the partial differential Navier-Stokes equations. The final solution is then found in the form of a power series. Using this technique, Burns and Parkes [33] developed a solution for the viscous flow through axially symmetric pipes and symmetrical channels with sinusoidal walls. They assumed that the Reynolds number is small enough for the Stokes flow approximation and found the stream functions in the form of a Fourier series for sinusoidal wall shapes. Later, Manton [34] extended the method of Burns and Parkes [33] to the tubes with arbitrary wall shapes. Although accurate results can be obtained for axial velocity using the perturbation method, it has been shown that there is a large error associated with the radial velocity [35]. The radial velocity is,

however, much smaller than the axial velocity, and thus it does not significantly affect the eventual pressure drop [35].

In all above mentioned studies, the 2D assumption has been made to simplify the Navier-Stokes and the continuity equations. Investigation of the flow and pressure drop for three dimensional problems are uncommon in the literature. Wild et al. [36], used the lubrication theory to calculate the pressure distribution in an elliptical tube, whose cross-sectional area varies slowly in longitudal direction. They also used a perturbation technique, up to the first order, to approximate the axial and transversal velocity distributions. Their results were reported in a complicated series form. Recently, Lauga et al. [14] investigated three-dimensional flow in slowly varying planar geometries. Using the perturbation theory, they showed that the fluid velocity has three components even if the channel cross-section varies only in one dimension.

Our literature review indicates the need for a compact analytical model that can accurately predict the pressure drop of the flow in rectangular cross-sections. The present work is motivated by practical applications of stream-wise-periodic microchannels and lack of data for low Reynolds number fully developed flow in varying microchannels of rectangular cross-sections. Starting from the general model of Bahrami et al. [37,38] for fully developed flow in arbitrary cross-section microchannels and using the lubrication approximation at each axial location, compact models have been developed to predict the pressure drop in slowly linear and sinusoidal varying microchannels. The proposed models are then examined for two limiting two-dimensional cases of: (1) very wide and (2) very high channels. Experimental and numerical data from the literature and an independent numerical

investigation in this work have been used to evaluate the accuracy of the proposed models.

2 Theory

The flow of a Newtonian fluid through varying cross-section channels is governed by the Navier-Stokes equations. Under certain geometrical and flow conditions which usually occur in microchannels, the Navier-Stokes equations can be reduced to the Poisson's equation. The flow requirement is that the viscous forces dominate the inertia forces. A quantitative statement of this criterion that can be obtained through scale analysis is [39]

$$Re_{\mathcal{L}}\left(\frac{\mathcal{L}}{\lambda}\right) \ll 1 \qquad (1)$$

where $\mathcal{L}$ is a characteristic length that corresponds to the variation of the cross-section, λ is the wavelength of the streamwise channel, and $Re_{\mathcal{L}} = \rho \bar{u} \mathcal{L}/\mu$ is the Reynolds number based on $\mathcal{L}$. This criterion indicates that either the Reynolds number is too small or the channel wave length is too large to consider Stokes flow. For the geometric condition, a slowly varying cross-section microchannel is assumed. Effects of the ends, edges, and other discontinuities, which have only local influence on the flow are disregarded here.

Selection of the characteristic length is an arbitrary choice and will not affect the final solution. However, an appropriate length scale leads to more consistent results, especially when general cross-section is considered. A circular duct is fully described with its diameter. For non-circular cross-sections, the selection is not as clear; many textbooks and previous works have employed the hydraulic diameter as the characteristic length. Yovanovich [40, 41] introduced the square root of area as a characteristic length scale for heat conduction and convection problems. Later, Muzychka and Yovanovich [42] proposed the use of $\sqrt{A}$ for the fully-developed flow in non-circular ducts. Bahrami et al. [37, 38] showed that the square root of area appears in the solution of fully-developed flow in non-circular ducts. They also compared both D_h and $\sqrt{A}$ and observed that using $\sqrt{A}$ as the characteristic length scale results in similar trends in Poiseuille number for microchannels with a wide variety of cross-sections. Therefore, in this study, $\sqrt{A}$ is selected consistently as the length scale throughout the analysis.

Starting from the exact solution of laminar fully developed flow in elliptical tubes, and using the Saint-Venant's approach [48], Bahrami et al. [37] developed a relationship for the Poiseuille number, $fRe_{\sqrt{A}}$ in the form of general geometrical functions as follows

$$fRe_{\sqrt{A}} = 32\pi^2 I_p^* \frac{\sqrt{A}}{\Gamma} \qquad (2)$$

where $I_p^* = I_p/A^2$ is specific polar moment of inertia, A and Γ are the cross-sectional area and perimeter, respectively. They [38] demonstrated that Eq. (2) can be used for other cross-sectional geometries including: rectangle, hyper-ellipse, trapezoid, sine, square duct with two rounded corners, circular sector, and moon-shaped channels with reasonable accuracy (within 8%).
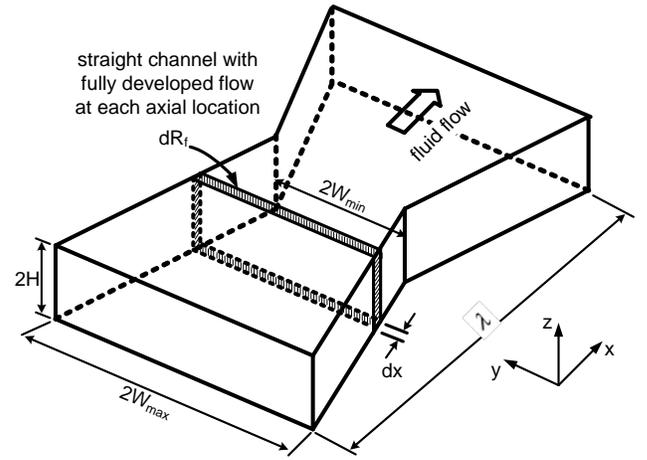

Figure 1. Schematic of a module of slowly varying microchannel of rectangular cross-section.

Defining flow resistance with an electrical analogy in mind, $R_f = \Delta p/Q$, the general solution of $fRe_{\sqrt{A}}$, Eq. (2) for rectangular cross-section channel depicted in Figure 1, and the lubrication approximation, flow resistance at each axial position can be found from the following relationship

$$dR_f = \frac{\pi^2 \mu}{12W^4(x)} \frac{1 + \varepsilon(x)^2}{\varepsilon(x)^3} dx \qquad (3)$$

where $\varepsilon(x)$ is the local aspect ratio at each axial position. Total flow resistance can then be obtain by integrating Eq. (3) over the entire length of the channel

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$$R_f = \frac{\pi^2 \mu}{12} \int_0^\lambda \frac{1 + \varepsilon(x)^2}{W^4(x)\varepsilon(x)^3} dx \qquad (4)$$

Equation (4), is a general relationship and can be applied to any wall shape. Two wall geometries, linear and sinusoidal, with constant height of $2H$, are investigated in present study. Figure 1, depicts the schematic of a module of converging-diverging channel with slowly varying wall. A streamwise-periodic channel with a linearly varying wall can be defined as

$$W(x) = W_0 - \frac{4\delta}{\lambda}\left(x + \frac{\lambda}{4}\right); \qquad -\frac{\lambda}{2} < x < 0$$
$$W(x) = W_0 + \frac{4\delta}{\lambda}\left(x - \frac{\lambda}{4}\right); \qquad 0 < x < \frac{\lambda}{2} \qquad (5)$$

where $W_0 = (W_{max} + W_{min})/2$ is the mean width of the channel, $\delta = (W_{max} - W_0)$ is the difference between the maximum and the mean widths and λ is the wave length of the streamwise-periodic channel. Substituting Eq. (5) into Eq. (4) yields

$$R_f^* = \frac{R_f}{R_{f0}} = \frac{2\xi \varepsilon_0^2 + (1 - \xi^2)^2 \ln\left(\frac{1 + \xi}{1 - \xi}\right)}{2\xi(1 - \xi^2)^2(1 + \varepsilon_0^2)} \qquad (6)$$

where $\varepsilon_0 = H/W_0$ is the mean aspect ratio of the channel, and $\xi = \delta/W_0$ is called the "deviation parameter" and shows the maximum variation of the wall from a straight channel of width W_0. The dimensionless flow resistance in Eq. (5), R_f^*, gives the ratio of the pressure drop in a varying cross-sectional channel over the straight channel with mean width. For a sinusoidal wall which is defined as

$$W(x) = W_0 - \delta\cos\left(\frac{2\pi x}{\lambda}\right) \qquad (7)$$

dimensionless flow resistance can be obtained from the following relationship

$$R_f^* = \frac{2(1 - \xi^2)^2 + (2 + \xi^2)\varepsilon_0^2}{2(1 - \xi^2)^{5/2}(1 + \varepsilon_0^2)} \qquad (8)$$

In many of the papers found in the literature, Poiseuille number, $Po_{\sqrt{A}} = fRe_{\sqrt{A}}$ is usually used instead of the flow resistance. The following relationship can be used to convert the dimensionless flow resistance to the Poiseuille number

$$fRe_{\sqrt{A}} = \frac{4\pi^2(1 + \varepsilon_0^2)}{3\sqrt{\varepsilon_0}(1 + \varepsilon_0)} R_f^* \qquad (9)$$

For rectangular microchannels, two asymptotes can be recognized, i.e. (1) the very narrow, $\varepsilon \ll 1$, and the very tall, $\varepsilon \gg 1$, channels. Figure 2, depicts these two asymptotes for a linearly varying cross-section. These two asymptotic problems are two-dimensional and pressure drop can be calculated independently either via lubrication theory or perturbation technique. Since lubrication approximation is used for the three dimensional flow, the same technique is applied here.
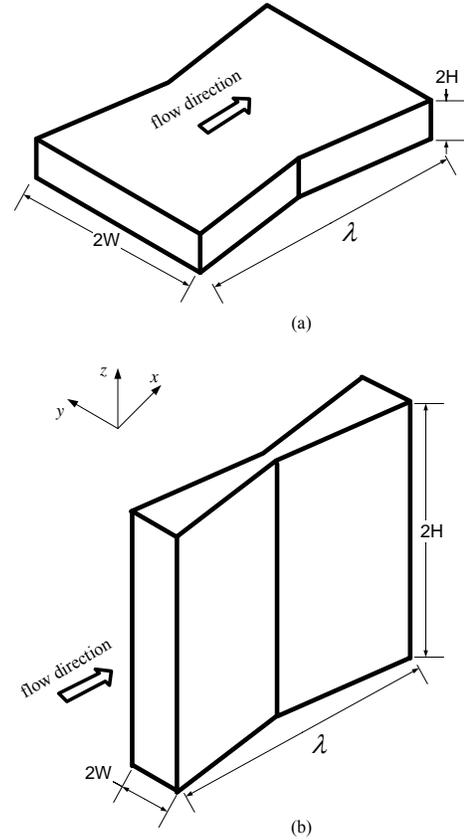

(a)

(b)

Figure 2. Schematic of the two asymptotic geometries for rectangular cross-section channels with varying cross-section (a) very narrow, and (b) very tall channel.

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2.1 Very narrow channels, $\varepsilon \ll 1$

Very wide channels (parallel plates) can usually be found in micromixers, microreactors and lab-on-chip devices [11-13]. Since $H \ll W$, the flow can be assumed to be two-dimensional and the influence of the wall in the y direction is negligible. The velocity profile between two parallel plates at each axial location has the following parabolic form

$$u(x,z) = \frac{3}{2}\bar{u}(x)(1 - \eta^2) \qquad (10)$$

where $\eta = z/H$ $\bar{u}(x) = Q/4HW(x)$ is the average velocity at each cross-section, and Q is the volumetric flow rate. Using the flow resistance definition, $R_f = \Delta P/Q$

$$R_f = \frac{3\mu}{2H^3}\int_0^\lambda \frac{dx}{W(x)} \qquad (11)$$

For linear varying cross-section defined by Eq. (5), dimensionless flow resistance can be obtained from the following relationship

$$R_f^* = \frac{R_f}{R_{f0}} = \frac{\ln\left(\frac{1+\xi}{1-\xi}\right)}{2\xi}; \qquad R_{f0} = 3\mu\lambda/2\varepsilon_0^3 W_0^4 \qquad (12)$$

where $\xi = \delta/W_0$ is the deviation parameter. Dimensionless flow resistance for a very narrow channel with sinusoidal wall defined by Eq. (7) can be obtained from

$$R_f^* = \frac{1}{(1-\xi^2)^{1/2}} \qquad (13)$$

The following asymptotic relationship can be used to convert the flow resistance to $fRe_{\sqrt{A}}$ for very narrow channels

$$fRe_{\sqrt{A}} = \frac{6\ln\left(\frac{1+\xi}{1-\xi}\right)}{2\xi}; \qquad \text{linear}$$

$$\qquad (14)$$

$$fRe_{\sqrt{A}} = \frac{12}{\sqrt{\varepsilon_0}\left(1-\xi^2\right)^{1/2}}; \qquad \text{sinusoidal}$$

For both linear and sinusoidal varying cross-sectional channels, at the limiting case when $\xi \to 0$, i.e. a channel with a constant width of W_0, $R_f^* \to 1$. Note that for $\xi = 1$, the flow is blocked and $R_f^* \to \infty$.

Note that, Eqs. (12) and (13) can also be obtained from Eqs. (6) and (8), respectively by letting $\varepsilon_0 \to 0$.
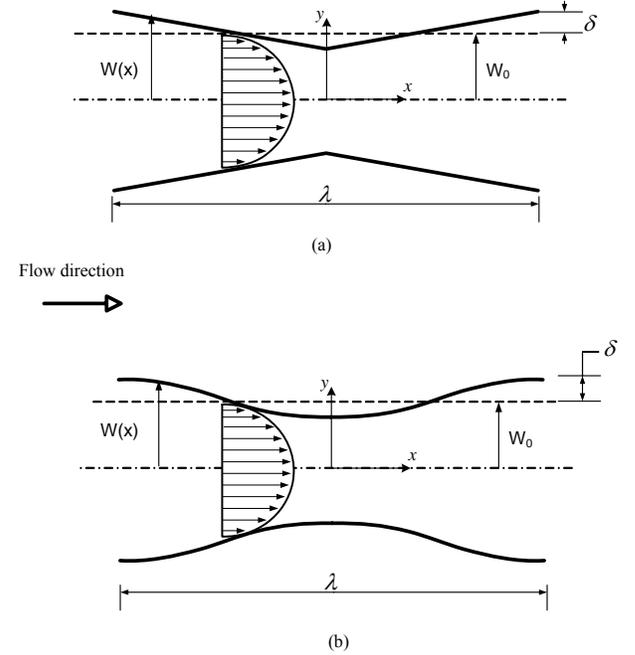

Flow direction

Figure 3. Geometrical parameters of a two-dimensional periodic slowly varying channel (a) linear and (b) sinusoidal variation.

2.2 Very tall channels, $\varepsilon \gg 1$

This channel geometry appears in the analysis of fractures in petroleum reservoirs or underground disposal of radioactive wastes [35]. Using the lubrication approximation at each axial location, the total flow resistance in a slowly varying cross-section channel yields

$$R_f = \frac{3\mu}{16H}\int_0^\lambda \frac{dx}{W^3(x)} \qquad (15)$$

For a linearly varying channel, dimensionless flow resistance can be obtained by substituting the variation of width from Eq. (5) into Eq. (15). Hence

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$$R_f^* = \frac{R_f}{R_{f0}} = \frac{1}{(1-\xi^2)^2}; \quad R_{f0} = 3\mu\lambda/2\varepsilon_0 W_0^4 \quad (16)$$

and for the sinusoidal wall

$$R_f^* = \frac{2+\xi^2}{2(1-\xi^2)^{5/2}} \quad (17)$$

Again, $R_f^* \to 1$ when the deviation parameter, $\xi \to 0$ and the channel is blocked when $\xi = 1$.

To convert the flow resistance to the Poiseuille number, the following asymptotic relationship can be used

$$fRe_{\sqrt{A}} = \frac{12\sqrt{\varepsilon_0}}{(1-\xi^2)^2}; \quad \text{linear}$$

$$\quad (18)$$

$$fRe_{\sqrt{A}} = \frac{6\sqrt{\varepsilon_0}(2+\xi^2)}{(1-\xi^2)^2}; \quad \text{sinusoidal}$$

Note that, Eqs. (16) and (17) can also be obtained from Eqs. (6) and (8), respectively, by letting $\varepsilon_0 \to \infty$.

3 Numerical analysis

To validate the proposed compact models, a numerical investigation has been conducted, using commercial finite volume base software, FLUENT [45]. Since the fully developed flow is usually achieved after two or three cycles [6, 26], numerical analysis is performed for a single module of converging-diverging channel with both linear and sinusoidal wall shapes. A range of aspect ratio, ε_0, deviation parameter, ξ, and Reynolds number, $Re_{\sqrt{A}}$ are considered which are listed in Table 2.

The working fluid is considered to be Newtonian with constant properties. Laminar flow with the Reynolds number ranging from 0.01 to 50 is examined to assure that the flow is laminar and stable [13, 16 and 46]. A structured, mapped mesh is used to discretize the numerical domain. A segregated method has been employed with a second order upwind scheme to solve the continuity and momentum equations. The SIMPLEC algorithm is used for pressure-velocity coupling. Large values of under-relaxation factor (0.8-0.95) are used to increase the convergence speed. Convergence is reached when the residuals for all equations become less than 10^{-5}.

Table 2. Numerical parameters

Parameter	Value
ε_0	0.1-1
ξ	0-0.5
$Re_{\sqrt{A}}$	0.01-50

No–slip boundary condition is applied to the module walls. Since the flow reaches the fully developed condition after a few cycles, periodic boundary condition [6] is applied at the module inlet and outlet. The assumption of periodicity implies that the velocity components repeat themselves in the flow direction, i.e. $\mathbf{u}(x,y,z) = \mathbf{u}(x+\lambda,y,z)$ [6], where $\mathbf{u}(x,y,z)$ is the velocity vector. For a periodic condition, the cross-sectional pressure distributions at x and $x+\lambda$ are identical in shape, but the pressure level decreases downstream. Pressure drops between two cycles with the periodic boundary condition are identical [6], i.e. $\Delta P = P(x+\lambda,y,z) - P(x,y,z) = P(x+2\lambda,y,z) - P(x+\lambda,y,z)$. Hence, the local pressure can be decomposed into two components: (1) periodic, $P'(x,y,z)$ and (2) linearly-varying, ηx [6]

$$P(x,y,z) = -\eta x + P'(x,y,z) \quad (19)$$

where $P'(x,y,z) = P'(x+\lambda,y,z)$ and η is the pressure drop per unit length. Using Eq. (19) the pressure drop of a stream-wise periodic module can be obtained from the following relationship

$$\Delta P_{module} = \lambda\eta \quad (20)$$

To ensure the independency of the present solution from the mesh resolution, a mesh refinement analysis has been conducted for all calculations. The value of the dimensionless flow resistance, R_f^* has been monitored since the velocity profile in any cross-section has not changed significantly.

4 Results and discussion

We now compare the results given by the present analytical study to numerical and experimental data obtained from earlier works. Numerical data of Niceone and Nobile [19] and Bahaidarah [24], and experimental data of Nishimura et al. [17] are used for two-dimensional

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sinusoidal wall. These previous investigations [17, 19 and 24] used similar geometrical parameters and cover a wide range of Reynolds numbers. However, only Reynolds numbers below 100 are considered in this study.

In Figure 3 the variations of dimensionless flow resistance, fRe_{W_0}, with respect to the Reynolds number for two-dimensional sinusoidal wall shape is plotted. We used the mean width, W_0, as a characteristic length scale, to be consistent with the notation used in the literature [17, 19 and 24]. To convert dimensionless flow resistance to the Poiseuille number, $fRe_{W_0} = 12R_f^*$ is used. Results obtained from the present analytical model, Eq. (17) agree well with the data, especially for Reynolds numbers lower than 50. The parameter fRe_{W_0}, is known to be constant for low Reynolds numbers [46]. When the Reynolds number increases, as a result of instabilities in the flow [47], deviations from the analytical model become more significant.
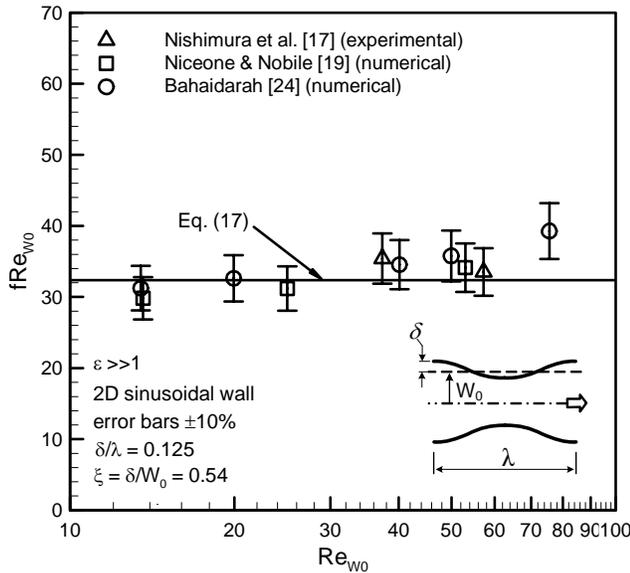

Figure 4. Variation of fRe_{W_0} with Reynolds number for two-dimensional sinusoidal channel.

The variation in fRe_{W_0} with respect to the Reynolds number for a two-dimensional linearly varying wall is plotted in Figure 5. Since no data have been found in the literature for low Reynolds number flow in linearly varying channels, an independent numerical analysis is carried out for three typical values of the deviation parameter, $\xi = \delta/W_0 = 0.1$, 0.3 and 0.5. As shown, numerical data is independent of the Reynolds number for $Re_{W_0} < 50$. For higher Reynolds number, due to instabilities in the flow, values of the flow resistance are no longer remaining constant and increase with the Reynolds number.
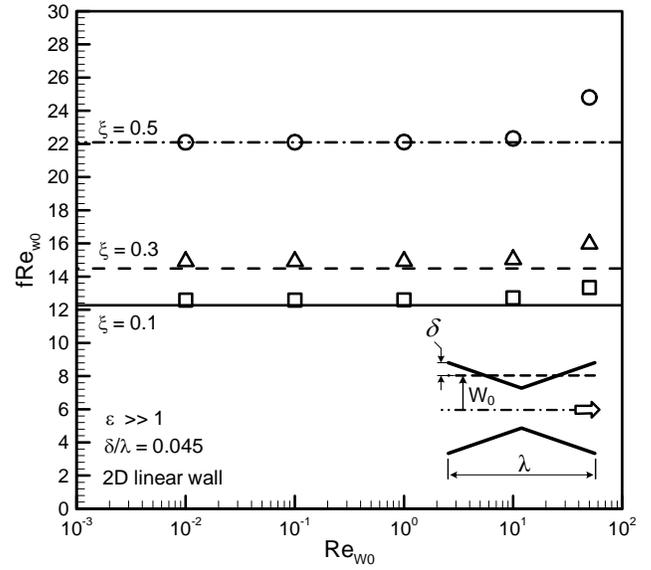

Figure 5. Variation in dimensionless flow resistance with Reynolds number for two-dimensional linear wall shapes. Square, delta and circular symbols correspond to the numerical data of the present work for typical values of the deviation parameter: 0.1, 0.3 and 0.5, respectively.

Effect of the deviation parameter, ξ on fRe_{W_0} is plotted in Figure 6. Numerical data from the numerical study of Mahmud et al. [15] is plotted with the proposed model for the sinusoidal wall case. Independent numerical results of the present study are also provided in Figure 6 for both linear and sinusoidal walls. It can be seen that, present models under predict the Poiseuille number, in general. The underpredictio is due to the use of the lubrication approximation which neglects the effects of pressure variation in transversal direction. Similar results have been reported by Sisavath et al. [35] for creeping flow in slowly varying pipes. Figure 6 shows that fRe_{W_0} raises with the increase of ξ. This is due the fact that for higher values of ξ, the channel lateral area is larger and as a result more shear stress is present in the flow. Similarly, since the wall length is smaller for the linear case, the flow resistance of the sinusoidal wall is higher for a given value of ξ.
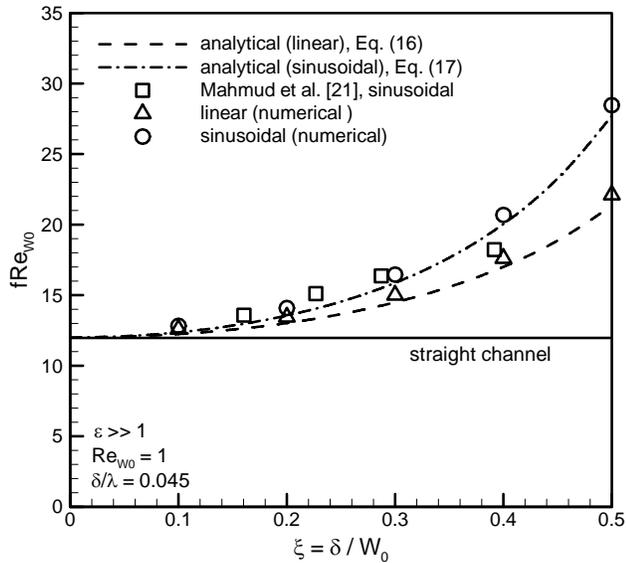
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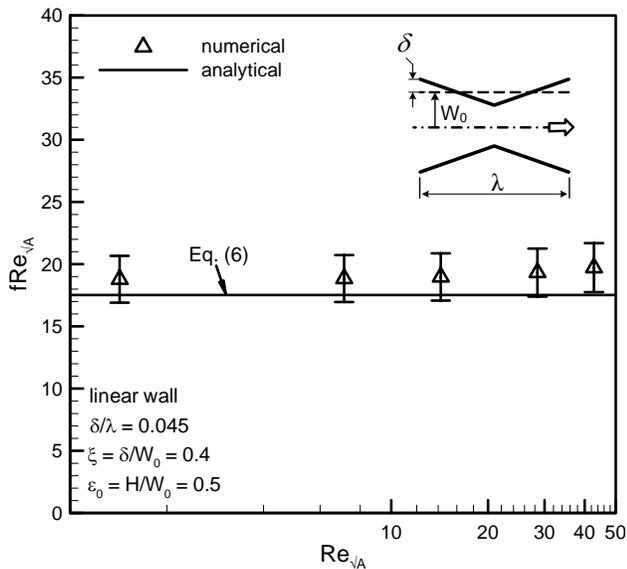**Figure 6. Effect of the deviation parameter, ξ on fRe_{W_0}.**

Figure 7. Variation of the Poiseuille number, $fRe_{\sqrt{A}}$, with the Reynolds number for linearly varying channels of rectangular cross-section; error bars indicate $\pm 10\%$ error.

Due to a lack of experimental and/or numerical data in the literature for the pressure drop of slowly varying microchannels of rectangular cross-section, we performed an independent numerical analysis to assess the validity of Eqs. (6) and (8). Poiseuille number, $fRe_{\sqrt{A}}$ is investigate instead of the flow resistance and Eq. (9) is used to convert the dimensionless flow resistance to $fRe_{\sqrt{A}}$. As

expected, the Poiseuille number, $fRe_{\sqrt{A}}$ is not a function of the Reynolds number for stokes flow. Figures 7 and 8 show the variation of $fRe_{\sqrt{A}}$ with respect to the Reynolds number for two typical values of the channel aspect ratio, $\varepsilon_0 = 0.5$, and deviation parameter, $\xi = 0.4$. The Reynolds number for the rectangular cross-section is defined using the square root of area as a characteristic length. As shown, although the analytical models for both sinusoidal and linearly varying channels underpredict the Poiseuille number, they still capture the numerical data within $\pm 10\%$.
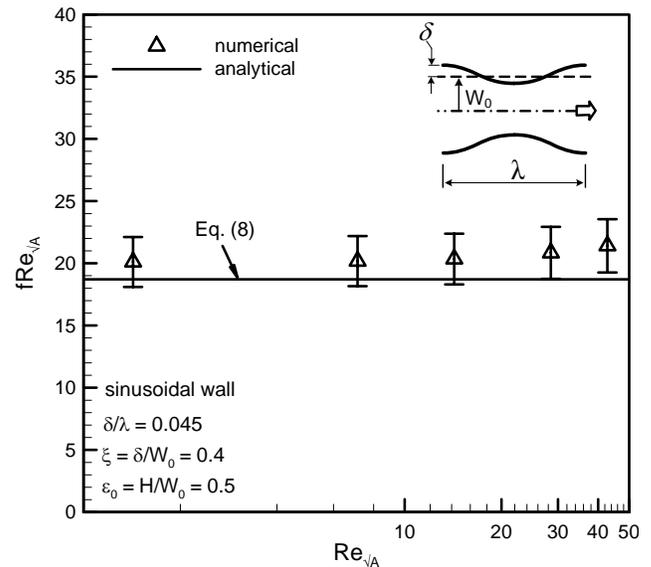

Figure 8. Variation of the Poiseuille number, $fRe_{\sqrt{A}}$, with the Reynolds number for sinusoidal channels of rectangular cross-section; error bars indicate $\pm 10\%$ error.

Figure 9 shows the variation of $fRe_{\sqrt{A}}$ with respect to the deviation parameter, ξ, for a rectangular cross-section channel for the typical values of the aspect ratio $\varepsilon_0 = 0.5$ and $Re_{\sqrt{A}} = 1$. Both sinusoidal and linear varying cross-sectional channels are considered in Figure 9. Analytical and numerical data show that the flow resistance increases with the deviation parameter ξ. Similar to the two-dimensional cases, this increase in the flow resistance can be attributed to the increase in the lateral surface area. As can be seen, the analytical model and numerical data show that the flow resistance of a sinusoidal wall is higher than a linearly varying channel.

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Table 3. Values of dimensionless flow resistance, R_f^* calculated from the analytical model versus numerical data.

	Linear, Eq. (6) $\xi = 0.4; \delta/\lambda = 0.045; Re_{\sqrt{A}} = 1$			Sinusoidal, Eq. (8) $\xi = 0.3; \delta/\lambda = 0.045; Re_{\sqrt{A}} = 1$		
$\varepsilon = H/W_0$	Numerical	Analytical	Error* %	Numerical	Analytical	Error* %
0.1	1.043	1.063	1.87	1.019	1.051	3.09
0.3	1.122	1.089	2.94	1.092	1.071	1.94
0.5	1.212	1.131	6.71	1.184	1.103	6.80
0.7	1.289	1.177	8.67	1.245	1.139	8.56
0.9	1.343	1.219	9.22	1.291	1.171	9.26
2	1.458	1.346	7.70	1.376	1.268	7.86
4	1.451	1.396	3.80	1.358	1.307	3.79
6	1.456	1.408	3.33	1.331	1.315	1.14
8	1.453	1.412	2.82	1.312	1.319	0.52
10	1.454	1.414	2.75	1.299	1.320	1.67

$* \ Error\% = \dfrac{\left|R_{f,num.}^* - R_{f,anal.}^*\right|}{R_{f,num.}^*} \times 100$
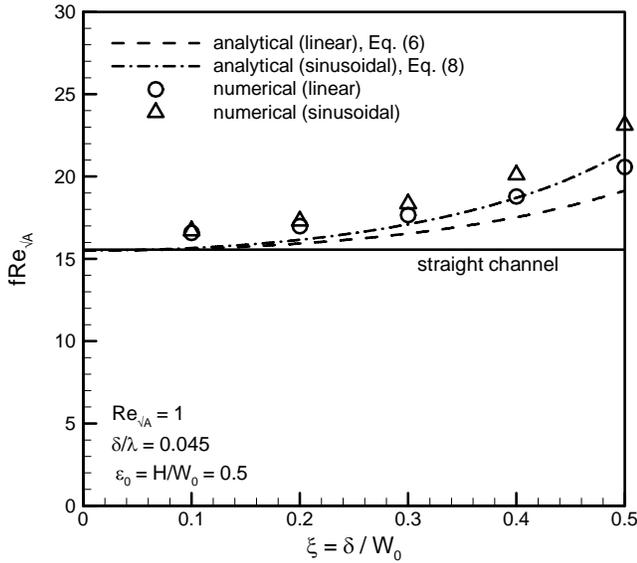

Figure 9. Dimensionless flow resistance as a function of the deviation parameter, ξ.

In a rectangular cross-section channel, the flow resistance depends on the channel aspect ratio, $\varepsilon_0 = H/W_0$, as well. Values of dimensionless flow resistance, R_f^* as a function of the channel aspect ratio are listed in Table 3. As can be seen, analytical results capture the values of the numerical data within the accuracy of $\pm 10\%$, for both linear and sinusoidal wall shapes. For small aspect ratios, the effect of the side walls on the flow is negligible. Hence, the pressure drop is only due to the frictional resistance of the upper and lower walls, i.e.

parallel plates. Increasing the aspect ratio, shear stress due to side walls can no longer be neglected and thus the flow resistance is larger. For very high aspect ratios, i.e. $\varepsilon_0 \to \infty$, upper and lower walls apply negligible shear stress on the flow. Hence, flow resistance remains constant which means by increasing the aspect ratio, pressure drop per module does not change.

Variation of the Poiseuille number, $f Re_{\sqrt{A}}$, with respect to the aspect ratio for linear and sinusoidal walls is plotted in Figures 10 and 11, respectively. To better show the trend of the analytical models, two asymptotes of very narrow and very tall are also included in Figure 10 and 11. Note that, Eqs. (6) and (8) recover Eq. (14) when $\varepsilon_0 \to 0$. For very high aspect ratios i.e. $\varepsilon_0 \to \infty$, Eqs. (6) and (8) give Eq.(18).

5 Summary

Pressure drop of fully developed, laminar flow in slowly varying microchannels of rectangular cross-section is investigated. We use the "lubrication approximation" at each axial location and combine it with the general approximate model of Bahrami et al. [31] for arbitrary cross-section microchannels, to develop compact analytical models for the linear and sinusoidal wall shapes. Developed models are found to be only a function of geometrical parameters including, cross-section aspect ratio, $\varepsilon_0 = H/W_0$ and deviation parameter, $\xi = \delta/W_0$. In the limiting case for very small aspect ratios, the model gives the solution for two-dimensional problem of a very narrow channel. For very high aspect ratios, the models

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result in the solution of a two-dimensional problem of a slowly varying wall. The following results are also obtained through the present analysis:

- As reported in the literature for two dimensional flow, the lubrication approximation for rectangular cross-section generally under predicts the values of $fRe_{\sqrt{A}}$ or equivalently flow resistance, R_f^* compared to the experimental and numerical data. However, it is observed that present models capture the trend of numerical and experimental data with accuracy within 10%.
- For both two-dimensional and rectangular cross-section channels, $fRe_{\sqrt{A}}$ increases with the deviation parameter, $\xi = \delta/W_0$. This increase is attributed to the increase in the lateral channel area.
- $fRe_{\sqrt{A}}$ of a sinusoidal wall is observed to be larger than a linear varying wall since the sinusoidal varying channel has larger lateral area.
- It is observed that the flow resistance, R_f^* increases with the channel aspect ratio, ε_0, and becomes constant for very high aspect ratios. At two limiting cases, the models tend to the two-dimensional problems of very narrow and very tall channels.

An experimental study is to evaluate the proposed models is the focus of current work. Preliminary results show good agreement between experimental data and the results of the present study.

6 Acknowledgments

Nomenclature

A	=	microchannel cross-sectional area, m^2
$fRe_{\sqrt{A}}$	=	Poiseuille number, -
H	=	channel half height, m
I_p^*	=	specific polar momentum of inertia, I_p/A^2
Q	=	volumetric flow rate, m^3/s
$Re_{\sqrt{A}}$	=	Reynolds number, $\rho Q/\mu\sqrt{A}$
R_f	=	flow resistance, $\Delta p/Q$
R_f^*	=	dimensionless flow resistance, $R_f/R_{f,0}$
$R_{f,0}$	=	flow resistance of a straight channels

W_0	=	microchannel mean half width, m

Greek

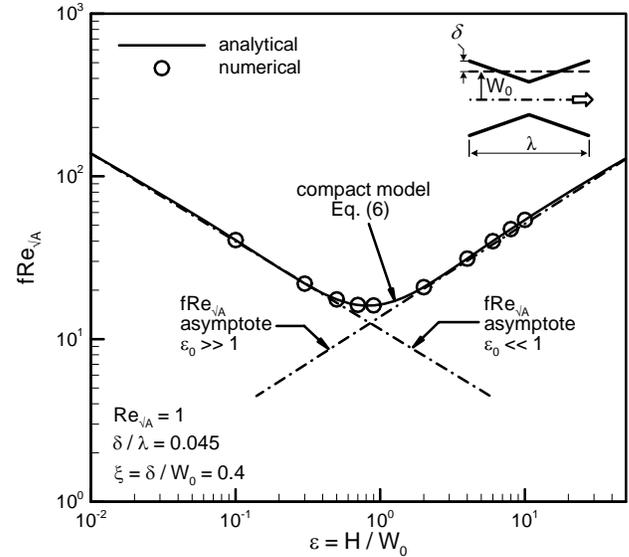|		
δ	=	deviation of width from the mean width, m
ε	=	channel aspect ratio, H/W
Γ	=	channel cross-section perimeter, m
λ	=	Length of module, m
μ	=	fluid viscosity, $kg/m.s$
ξ	=	deviation parameter, δ/W_0

Figure 10. Poiseuille number, $fRe_{\sqrt{A}}$ as a function of aspect ratio, linear wall.
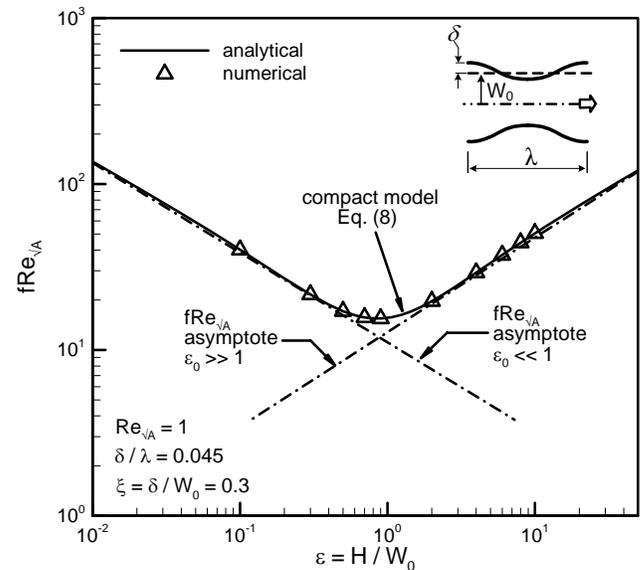

Figure 11. Poiseuille number, $fRe_{\sqrt{A}}$ as a function of aspect ratio, sinusoidal wall.

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